

# An Operational Approach to the Acoustic Analogy Equations

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## 1. Introduction

Great progress has been made in the last sixty years in the study of the important problem of noise generated by the interaction of flow with stationary or mobile bodies such as occurs in jets, rotating blade propulsion machinery (propellers, turbopans helicopter rotors) and last but not least in aircraft at all ranges of flight and speed. An important part of this progress was based on a rigorous theory known as the Acoustic Analogy initiated by Sir James Lighthill in (Lighthill, 1952) and (Lighthill, 1954). Lighthill considered a free flow, as for example with a jet engine, and the nonstationary fluctuations of the stream represented by a distribution of quadrupole sources in the same volume. The flow parameters such as the surface pressure and the Lighthill tensor  $T_{ij}$  are assumed known from solving the aerodynamic problem in the region of sound generation or furnished by measurements. For the first time, this revealed a clear distinction between Aerodynamic Theory, meant to determine mainly the aerodynamic parameters as the lift and damping on the moving object (and also supplying the data for the noise determination) and the Aeroacoustic Theory needed for studying the noise produced, generally at large distances, by the flying (or moving) objects. A primary aim of the following is to show that by using an operational calculus based on the multidimensional Fourier Transform all the theory involved in obtaining the Ffowcs Williams-Hawkings formula (Ffowcs Williams and Hawkings, 1969) can be performed using only classical mathematical analysis.

Curle's contribution (Curle, 1955) is a formal solution of the Acoustic Analogy which takes stationary hard surfaces into consideration. The theory developed by Ffowcs Williams and Hawkings (FW-H) (Ffowcs Williams and Hawkings, 1969) is valid for aeroacoustic sources in relative motion with respect to a hard surface, as is the case in many technical applications for example in the automotive industry or in air travel. The calculation involves quadrupole, dipole and monopole terms. An important point is that FW-H theory, developed in (Ffowcs Williams and Hawkings, 1969) assumed that the boundary surface coincides with the physical body surface and is impenetrable. In both Aerodynamic and Aeroacoustic theories the domain was the same: the infinite air domain external to the moving body.

When the Aeroacoustic Theory was developed by Lighthill, Curle, Ffowcs Williams and Hawkings there were not a lot of experimental or theoretical data to be used as input to their aeroacoustic theoretical work. For this reason, they derived mainly qualitative results

which were quite useful in guiding many significant acoustic experiments and in designing low noise propulsion machinery.

The situation has changed dramatically in the last 20 years. The rapid growth in high speed digital computer technology, the availability of turbulent flow simulation codes as well as high quality measured fluid dynamic data and advances in the theory of partial differential equations, resulted in obtaining the needed data for many important problems of aeroacoustics. However, the development of reliable Computational Fluid Dynamics (CFD) methods made them also useful in the evaluation of the near-field aerodynamic parameters. Unfortunately, a fully CFD-based computational aeroacoustic methodology is so far too inefficient and beyond the capability of supercomputers of today.

To avoid these computational difficulties, the philosophy of approaching the Aeroacoustic problem has changed by introducing a surface  $S$  as a "permeable" control surface. The surface  $S$  is assumed to include inside, in the volume  $V$ , all the nonlinear flow effects and noise sources. This splitting of the problem into a linear problem for an infinite domain and a nonlinear setting for a bounded region allows the use of the most appropriate numerical methodology for each of them. In the bounded domain  $V$  the CFD methods or advanced measurement techniques will be used for obtaining the aerodynamic near-field and providing the data on the surface  $S$  needed for the external, infinite domain modelling. The analysis of the flow information inside  $V$  is, in general, expensive either using experiments or CFD. Therefore, it is advantageous to make the volume  $V$  as small as possible.

The FH-W equation involving a permeable surface is the proper model for determining the far-field pressure in the infinite domain. The case of permeable surfaces was analyzed by Ffowcs Williams in (Dowling and Ffowcs Williams, 1982) and (Crighton, et al.), by Francescantonio in (Francescantonio, 1997) who called it the KFWH (Kirchhoff FW-H) formula, by Pilon and Lyrintzis in (Pilon and Lyrintzis, 1997) calling it an improved Kirchhoff method and by Brentner and Farassat in (Brentner & Farassat, 1998). Besides the accessibility to the surface data the advantage of the methods using a permeable control surface is that the surface integrals and the first derivative needed can be evaluated more easily than the volume integrals and the second derivatives necessary for the calculation of the quadrupole terms when the traditional Acoustic Analogy is used.

The Acoustic Analogy approach and especially the theory based on the FW-H equation is the most widely used tool for deterministic noise sources. The beauty (and the power) of this model is that all the manipulations are completely rigorous without any *ad hoc* reasoning.

Besides the approach based on generalized functions, used in most papers approaching the FW-H formula, we note the work by Goldstein (Goldstein, 1976) where all the formulas are obtained starting with a generalized Green's formula. A similar approach was used in (Wu and Akay, 1992). However, the algebra involved in their construction is substantial.

In the second of the two reports by Farassat (Farassat, 1994),(Farassat, 1996), which covers the details of the mathematics used for the wave equation with sources on a moving surface, the author correctly claims that the Ffowcs Williams-Hawkings famous paper published in 1969 used a level of mathematical sophistication including multidimensional generalized functions (distributions) and differential geometry unfamiliar to most researchers and designers working in the field. Many people use Dirac's  $\delta$  (generalized) function starting with its integral definition. On the other hand, to learn about more complicated operations such as the derivative of a generalized function (distribution) we need a change of paradigm in the way we look at ordinary functions. For some people involved in practical applications this is not a simple task. The power of the theory of generalized functions stems from its

operational properties. Thus, for example, discontinuous solutions of linear equations using the Green's function are easily obtained by posing the problem in generalized function spaces. In the following we show that the theory connected with the FW-H formula can be made much simpler, without manipulating multidimensional generalized functions, by using an operational calculus based on the multidimensional Fourier Transform. The approach based on using the Fourier Transform preserves all the good operational properties of the generalized functions without the need to introduce a new sophisticated mathematical tool. The method was used previously by us (Homencovschi and Singler, 1999) for a direct introduction to the Boundary Element Method.

In the case of permeable surfaces an alternate method for solving the Aeroacoustic problem for the infinite external domain is based on the Kirchhoff formula for the wave equation. Due to its use in Aeroacoustic theory we included also in Section 6 the Kirchhoff formulation for the solution of the wave equation in the case of mobile surfaces. The proof is based again on 3-D Fourier Transform of discontinuous functions and the final formula includes the volume sources and the surface sources as well. As a comparison of the two approaches (that based on FW-K equation and that using Kirchhoff's equation) we notice that Kirchhoff's method requires less memory because fewer quantities on the control surface needed to be stored. On the other hand, Brentner and Farassat in (Brentner & Farassat, 1998) have shown that the FW-H equation is superior to the Kirchhoff formula for aeroacoustic problems because it is based on conservation laws of fluid mechanics rather than on the wave equation. Thus, the FW-H equation is valid even if the integration surface is in the nonlinear region being therefore more robust with the choice of control surface. Another advantage of the FW-H method is that it does not require computation of the normal derivatives on the permeable surface.

A comprehensive review of the use of Kirchhoff's method in computational aeroacoustics was given by Lyrintzis in (Lyrintzis, 1994). The same author reviewed the advances in the use of surface integral methods in aeroacoustics, including Kirchhoff's method and permeable Ffowcs-Williams Hawkins methods in (Lyrintzis, 2003). Morino in (Morino, 2003) addresses commonalities and differences between aeroacoustics and aerodynamics. A discussion about the acoustic analogy and alternative theories for jet noise prediction is the subject of (Morris and Farassat, 2002). Finally we note some interesting work about this subject included in the book edited by (Raman, 2009).

It is our hope that the elementary derivations included in this chapter will make the application of the FW-H equation more clear, avoiding in the future comments such as those generated by (Zinoviev and Bies, 2004) (See (Farassat, 2005), (Zinoviev and Bies, 2005), (Farassat and Myers, 2006), (Zinoviev and Bies, 2006)).

## 2. The equations of the acoustic analogy and their operational form

In order to apply the 3-D Fourier Transform to a certain physical variable this has to be defined in the whole space. Thus, to utilize the Fourier Transform to examine the sound field in a finite physical domain, it is necessary to imbed this domain within an infinite space. For example, in the case of studying the air motion around a finite body, the interior of the body is considered as an air-filled domain separated from the exterior, infinite domain by an impermeable surface,  $S_b$  enclosing the body. This introduces a first class of discontinuity surfaces of the motion between two air-filled regions. The second class contains the natural discontinuity surfaces  $S_s$  inside the flow domain as shock fronts, wakes, etc. Finally, to aid computations, it is often helpful to also define a virtual permeable surface,  $S_p$ , enclosing body along with the portion of the air domain where viscous effects and the nonlinear terms in

the Navier-Stokes equations are important. It is convenient to evaluate the aerodynamic field in this region numerically using CFD, due to the difficulties imposed by viscosity and nonlinear effects. These calculations furnish data describing the hydrodynamic state on the virtual permeable surface,  $S_p$ . In the domain exterior to the permeable surface the acoustical analogy is applied to predict the sound field. In other words, the field inside the virtual permeable surface is assumed to be strongly influenced by hydrodynamic effects, while the field in the external, infinite domain can be modeled according to the usual assumptions of linear acoustics.

### 2.1 The equations of the acoustic analogy

In the case where the whole space  $\mathbb{R}^3$  is filled by a compressible viscous fluid, containing several discontinuity surfaces, the flow inside each domain is governed by the following equations:

the continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0, \quad (1)$$

and the conservation of momentum in the form written by Lighthill in Refs. (Lighthill, 1952) and (Lighthill, 1954)

$$\frac{\partial (\rho u_i)}{\partial t} + c^2 \frac{\partial \rho}{\partial x_i} = - \frac{\partial T_{ij}}{\partial x_j}. \quad (2)$$

Here  $\rho$  is the density,  $c$  is the velocity of sound in the uniform medium,  $u_j$  is the component of fluid velocity in the direction  $x_j$  ( $j = 1, 2, 3$ ), and a repeated index implies a summation over these values, and,

$$T_{ij} = P_{ij} + \rho u_i u_j - c^2 \rho \delta_{ij},$$

is Lighthill's stress tensor. Also, we denoted by

$$P_{ij} = p \delta_{ij} + \mu \left( - \frac{\partial u_j}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + \frac{2}{3} \left( \frac{\partial u_\kappa}{\partial x_\kappa} \right) \delta_{ij} \right),$$

the compressive stress tensor,  $\delta_{ij}$  is the Kronicker delta function,  $p$  is the pressure and  $\mu$  is the viscosity.

**Remark 1.** *The acoustical analogy equations (1) and (2) are in fact the exact fluid flow equations in the form written by Lighthill. This means that if one solves these equations correctly for a problem satisfying the Lighthill assumptions, then one will get the correct answer to the aerodynamic and aeroacoustic problems simultaneously.*

In the case where inside the flow domain there are discontinuity surfaces of type  $S_s$  (as shock fronts) the conservation laws for mass and momentum yield the Rankine-Hugoniot type junction conditions

$$[\rho (u_n - v_n)] = 0 \quad (3)$$

$$[\rho u_\kappa (u_n - v_n) + P_{j\kappa} n_j] = 0. \quad (4)$$

We denote by square brackets the jump of its content across the discontinuity surface (see also formula (54) in Appendix A).  $u_n$  is the velocity projection on the normal to the surface,  $\mathbf{n}$ , and  $v_n$  is the normal velocity of the surface.

Finally, on solid (deformable) impermeable surfaces  $S_b$  inside the flow the obvious non-penetration condition proves true

$$u_n - v_n = 0. \quad (5)$$

## 2.2 The operational form of the acoustic analogy equations

Since the density  $\rho_0$  and pressure  $p_0$  at large distances (in the unperturbed fluid) are different from zero we introduce the perturbation of the density  $\rho' = \rho - \rho_0$  and the perturbation of the stress tensor as  $P'_{jk} = P_{jk} - p_0\delta_{jk}$  (where  $\delta_{jk}$  is the Kronecker delta) and will write equations (1) and (2) as

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0 \quad (6)$$

$$\frac{\partial (\rho u_j)}{\partial t} + c^2 \frac{\partial \rho'}{\partial x_j} = -\frac{\partial T'_{jk}}{\partial x_k}, \quad (7)$$

where

$$T'_{jk} = P'_{jk} + \rho u_j u_k - c^2 \rho' \delta_{jk}$$

According to the previous discussion, these equations are valid within the flow domains in the whole space. Assuming, as in Appendix A, that there is a discontinuity surface  $S$ , separating the inner domain  $D^{(i)}$  and the external domain  $D^{(e)}$ , by applying the Fourier transform to equations (6) and (7) and using the formulas (53) and (68) there results *the operational equations*

$$\frac{d \tilde{\rho}'}{dt} + ik_j \tilde{\rho} u_j = \int_S [\rho (u_n - v_n)] e^{-i\mathbf{k} \cdot \mathbf{y}} dS \quad (8)$$

$$\frac{d \tilde{\rho} u_j}{dt} + ik_j c^2 \tilde{\rho}' = -ik_k \tilde{T}'_{jk} + \int_S [\rho u_j (u_n - v_n) + P_{kj} n_k] e^{-i\mathbf{k} \cdot \mathbf{y}} dS \quad (9)$$

Here the overhead tilde denotes a Fourier Transform (see Appendix A),  $\mathbf{k}(k_1, k_2, k_3)$  is the wave vector and  $\mathbf{y}(y_1, y_2, y_3)$  is the position vector of the integration (source) point. The *operational* form of the conservation equations contains in the left-hand sides the Fourier transform of corresponding terms in (1) and (2) and in the right-hand sides the integrals accounting for the influence of the discontinuity surfaces. As was noted in Remark 1 in Appendix A, in the general case, in the right-hand side of equations (8) and (9) the sum of the contributions of all discontinuity surfaces will appear.

**Remark 2.** *The square brackets in equations (8) and (9) can also be written as  $[\rho_0 u_n + \rho' (u_n - v_n)]$  and  $[\rho u_k (u_n - v_n) + P'_{kj} n_j]$  respectively.*

## 2.3 Some particular cases

### 2.3.1 Shock-type discontinuity surfaces ( $S_s$ )

In the case of a discontinuity surface  $S_s$  (of shock-type) inside the fluid domain the junction conditions (3) and (4) will cancel out the integral terms in equations (8) and (9). Consequently the shock-type discontinuity surfaces are not introducing any supplementary terms in the operational equations.

### 2.3.2 Impermeable solid deformable surfaces ( $S_b$ )

In the case of a solid with a deformable and impermeable boundary surface the condition (5) will cancel out the integral term in the operational form of the continuity equation (8) and a part of the integral in the momentum equation (9). In this case, the Fourier transform of the traction  $\mathbf{P}=(P_1, P_2, P_3)$  of the surface on the fluid enters in the integral term of the momentum equations as  $P_j = P_{\kappa j} n_{\kappa}$ . The operational momentum equation now contains in the right-hand side the action of the solid surface  $S_b$  on the fluid flow.

## 3. Solution of the operational form of the acoustical analogy equations.

### 3.1 The case of a permeable surface ( $S_p$ )

Let  $S_p$  be a discontinuity surface of the flow variables inside the external fluid flow such that in the domain  $D^{(i)}$  we have  $p^{(i)} = p_0, \rho^{(i)} = \rho_0, \mathbf{v}^{(i)} = \mathbf{0}$ .

We write the system of equations (8), (9) as

$$\frac{d\tilde{\mathbf{w}}}{dt} = \tilde{\mathbb{A}}\tilde{\mathbf{w}} + \tilde{\mathbf{f}} \quad (10)$$

where

$$\tilde{\mathbf{w}}^T = (\tilde{\rho}', \tilde{\rho}u_1, \tilde{\rho}u_2, \tilde{\rho}u_3), \quad (11)$$

$$\tilde{\mathbb{A}} = \begin{bmatrix} 0 & -ik_1 & -ik_2 & -ik_3 \\ -c^2 i k_1 & 0 & 0 & 0 \\ -c^2 i k_2 & 0 & 0 & 0 \\ -c^2 i k_3 & 0 & 0 & 0 \end{bmatrix}, \quad (12)$$

$$\tilde{\mathbf{f}}_1 = \int_S [\rho_0 u_n + \rho' (u_n - v_n)] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \\ \tilde{f}_{j+1} = -ik_{\kappa} \tilde{T}_{j\kappa} + \int_S [\rho u_j (u_n - v_n) + P_j] e^{-i\mathbf{k}\cdot\mathbf{y}} dS, \quad j = 1, 2, 3. \quad (13)$$

The solution of equation (10) can be obtained by using the exponential  $\tilde{\mathbb{H}}(t)$  of the matrix  $\tilde{\mathbb{A}}$  as

$$\tilde{\mathbf{w}} = \int_{-\infty}^t \tilde{\mathbb{H}}(t-t') \tilde{\mathbf{f}}(t') dt' \quad (14)$$

The exponential of the matrix  $\tilde{\mathbb{A}}$  can be written as

$$\tilde{\mathbb{H}} = [\tilde{h}_{j\kappa}] \quad (15)$$

$$\tilde{h}_{1,1} = \cos(ckt), \quad \tilde{h}_{1,\kappa+1} = -\frac{ik_{\kappa} \sin(ckt)}{ck}, \quad \kappa = 1, 2, 3 \\ \tilde{h}_{j+1,1} = -\frac{ik_j c \sin(ckt)}{k}, \quad j = 1, 2, 3 \\ \tilde{h}_{j+1,\kappa+1} = \delta_{j\kappa} + \frac{k_j k_{\kappa} (\cos(ckt) - 1)}{k^2}, \quad j, \kappa = 1, 2, 3 \quad (16)$$

where  $k^2 = k_1^2 + k_2^2 + k_3^2 = |\mathbf{k}|^2$ . By introducing the function

$$\tilde{F}(\mathbf{k}, t) = \frac{d}{dt} \int_S \{ \rho_0 u_n + \rho' (u_n - v_n) \} e^{-i\mathbf{k}\cdot\mathbf{y}} dS \\ - ik_j \int_S \{ \rho u_j (u_n - v_n) + P_j \} e^{-i\mathbf{k}\cdot\mathbf{y}} dS + ik_j ik_{\kappa} \tilde{T}'_{j\kappa} \quad (17)$$

the components of the vector  $\tilde{\mathbf{w}}$  can be written as

$$\tilde{\rho}'(\mathbf{k}, t) = \int_{-\infty}^t \tilde{F}(\mathbf{k}, \tau) \frac{\sin(ck(t-\tau))}{ck} d\tau \quad (18)$$

$$\begin{aligned} \tilde{\rho}u_j(\mathbf{k}, t) = & - \int_{-\infty}^t ik_\kappa \tilde{T}_{j\kappa}(\mathbf{k}, \tau) d\tau + \int_{-\infty}^t d\tau \int_{S(\tau)} \left\{ \rho u_j (u_n - v_n) + P_j' \right\} e^{-i\mathbf{k} \cdot \mathbf{y}} dS \quad (19) \\ & + ik_j \int_{-\infty}^t \tilde{F}(\mathbf{k}, \tau) \frac{\cos(ck(t-\tau)) - 1}{k^2} d\tau \end{aligned}$$

The method given in this section has the advantage of furnishing the integral representations for the operational density and operational velocity as well. A simpler deduction of the representation of the operational density is given in the next section.

### 3.2 The equation for the operational density

By eliminating the Fourier transform of the momentum  $\tilde{\rho}u_j$  between equations (8) and (9) the operational equation satisfied by the density perturbation becomes

$$\frac{d^2 \tilde{\rho}'}{dt^2} + c^2 k^2 \tilde{\rho}' = \tilde{F}(\mathbf{k}, t). \quad (20)$$

The relationship (20) is the operational form of the nonhomogeneous wave equation. The general solution of the homogeneous wave equation in operational form can be written as

$$\tilde{\rho}'_h = A(k) \cos(ck t) + B(k) \sin(ck t)$$

and by using Lagrange's method of variation of parameters there results the same representation formula (18) for the operational density as the solution of equation (20).

### 3.3 The case of an impermeable surface ( $S_b$ )

In the case where the surface  $S_b$  is impermeable, the condition (5) cancels out some terms in formula (17). In this case the nonhomogeneous term is

$$\tilde{F}_b(\mathbf{k}, t) = \frac{d}{dt} \int_S \rho_0 u_n e^{-i\mathbf{k} \cdot \mathbf{y}} dS - ik_j \int_S P_j' e^{-i\mathbf{k} \cdot \mathbf{y}} dS + ik_j ik_\kappa \tilde{T}'_{j\kappa} \quad (21)$$

which coincides with the nonhomogeneous term in the operational form of the FW-H equation.

By introducing the new variables suggested by Francescantonio (Francescantonio, 1997), in the form modified in (Brentner & Farassat, 1998)

$$U_j = \left(1 - \frac{\rho}{\rho_0}\right) v_j + \frac{\rho u_j}{\rho_0} \quad (22)$$

$$L_j = P_j' + \rho u_j (u_n - v_n) \quad (23)$$

the nonhomogeneous term of the operational wave equation in the case of a permeable surface coincides with that corresponding to an impermeable case

$$\tilde{F}(\mathbf{k}, t) = \frac{d}{dt} \int_S \rho_0 U_n e^{-i\mathbf{k} \cdot \mathbf{y}} dS - ik_j \int_S L_j e^{-i\mathbf{k} \cdot \mathbf{y}} dS + ik_j ik_\kappa \tilde{T}_{j\kappa} \quad (24)$$

The terms  $U_j$  and  $L_j$  can be interpreted respectively as a modified velocity and a modified traction, which take into account the flow across  $S$ .

## 4. Determination of velocity

### 4.1 The lift component of velocity

The lift component of velocity is given by inverse Fourier transform of the term

$$\left(\widetilde{\rho u}_j\right)_L = ik_j ik_r \int_{-\infty}^t d\tau \int_{S_\tau} P'(\mathbf{y}, \tau) n'_r(\mathbf{y}, \tau) e^{-ik \cdot \mathbf{y}} \frac{\cos(ck(t-\tau)) - 1}{k^2} dS$$

Therefore, by using formula (73) we obtain

$$\left(\rho u_j\right)_L = \frac{\partial^2}{\partial x_j \partial x_r} \int_{-\infty}^t d\tau \int_{S_\tau} P' n'_r \frac{H(t-\tau-r/c)}{4\pi r} dS$$

where

$$r = |\mathbf{x} - \mathbf{y}| \equiv \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

$\mathbf{x}(x_1, x_2, x_3)$  being the position vector of the observation point.

## 5. Determination of density

Since the general permeable case can be studied by means of an equivalent impermeable case we shall determine the density in the case where the nonhomogeneous term is (21).

### 5.1 The case of sources on a surface

Consider the simpler case when the noise sources are on a rigid surface having only translation and rotation motions. Then,

$$\tilde{F}_S(\mathbf{k}, t) = \int_S Q(\mathbf{y}, t) e^{-ik \cdot \mathbf{y}} dS \quad (25)$$

where  $Q(\mathbf{y}, t)$  is the surface intensity. In this case we have

$$\tilde{\rho}'_S(\mathbf{k}, t) = \int_{-\infty}^t d\tau \int_{S_\tau} Q(\mathbf{y}, \tau) e^{-ik \cdot \mathbf{y}} \frac{\sin(ck(t-\tau))}{ck} dS \quad (26)$$

Hence,

$$\begin{aligned} \rho'_S(\mathbf{x}, t) &= \frac{1}{(2\pi)^3} \int e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \int_{-\infty}^t d\tau \int_{S_\tau} Q(\mathbf{y}, \tau) e^{-ik \cdot \mathbf{y}} \frac{\sin(ck(t-\tau))}{ck} dS \\ &= \int_{-\infty}^t d\tau \int_{S_\tau} Q(\mathbf{y}, \tau) dS \int \frac{\sin(ck(t-\tau))}{(2\pi)^3 ck} e^{i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})} d\mathbf{k} \\ &= \int_{-\infty}^t d\tau \int_{S_\tau} Q(\mathbf{y}, \tau) \frac{\delta(\tau - (t-r/c))}{4\pi c^2 r} dS \end{aligned} \quad (27)$$

Here the relationship (71) and the property  $\delta(-t) = \delta(t)$  have been used.

The only contribution in the last integral in formula (27) comes from the time  $\tau_e$  which is the solution of the equation

$$g(\tau, \mathbf{y}, t, \mathbf{x}) = 0 \quad (28)$$

where

$$g(\tau, \mathbf{y}, t, \mathbf{x}) = \tau - t + \frac{|\mathbf{x} - \mathbf{y}|}{c} \quad (29)$$

In other words, the value of the density at the observation point  $\mathbf{x}$  at the moment  $t$  is determined by the noise sources at the *emission (radiating) time*  $\tau_e$  on the *emission (radiating) surface*  $S_e \equiv S_{\tau_e}$ .

It is now necessary to consider the coordinate systems. Let a fixed point  $P_0$  reside on a material surface such as an airplane wing or a blade etc. We consider two coordinate frames:

a) A frame  $\eta$  fixed relative to the moving material surface. This is the frame used by a designer to describe the structure geometry for purposes of fabrication. The variable  $\eta$  will be called the Lagrangian variable of the point  $P_0$  and in the case supposed here (nondeformable material surface) is independent of time.

b) A coordinate frame fixed with respect to the undisturbed medium having the origin  $O$  (see Fig.1). The position of the observation point is given by the position vector  $\mathbf{x}$ . The position of the point  $P_0$  is described by the position vector  $\mathbf{y}(\eta, \tau)$ . The position vectors  $\mathbf{x}$  and  $\mathbf{y}$  will give the Eulerian coordinates of the observation and source terms, respectively. The formula

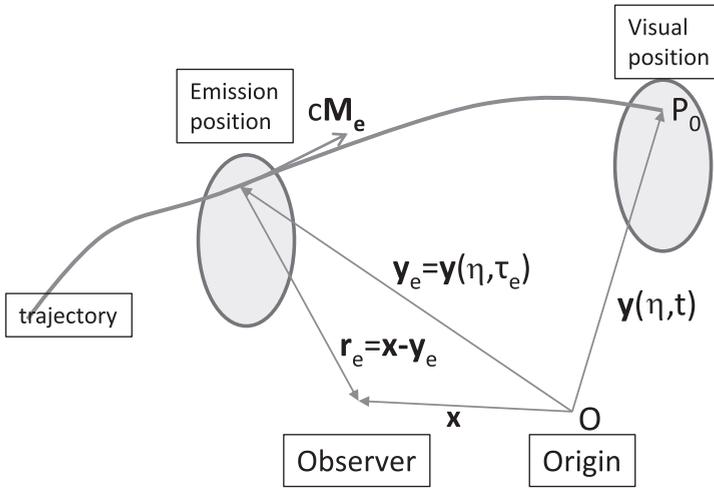


Fig. 1. Coordinate frame

$$\mathbf{y} = \eta + \int^{\tau} c\mathbf{M}(\eta, \tau') d\tau' \tag{30}$$

gives the connection between the Lagrangian and Eulerian coordinates of the point  $P_0$ . For  $\eta$  fixed the equation  $\mathbf{y} = \mathbf{y}(\eta, \tau)$  gives the trajectory of the point  $P_0$  and  $c\mathbf{M} = d\mathbf{y}/d\tau$  is the velocity of the source point with respect to the undisturbed medium (source convection velocity). Formula (30) can be viewed as a transformation between the Lagrangian and Eulerian coordinates of the point  $P_0$ . The inverse transformation will be denoted by  $\eta = \eta(\mathbf{y}, \tau)$ . In the case where the transformation involves only translations and rotations we have  $\det(\partial\mathbf{y}/\partial\eta) = \det(\partial\eta/\partial\mathbf{y}) = 1$ . Fig.1 shows the observer's position  $\mathbf{x}$  at time  $t$ , the emission (radiation) position of the material surface ( $S_e \equiv S_{\tau_e}$ ), the position of the same surface at the observation time ( $S_t$ ), the position vector  $\mathbf{y}_e$  at the emission time and at the observation time  $\mathbf{y}(\eta, t)$ , the trajectory of the point  $P_0$ , the convection velocity of the source at the emission moment  $c\mathbf{M}_e$ , the radiation vector  $\mathbf{r}_e = \mathbf{x} - \mathbf{y}_e$ , and the emission distance  $r_e = |\mathbf{r}_e|$ .

To change the integration variable in the last integral in formula (27) from  $\tau$  to  $g$ , we calculate

$$\frac{dg}{d\tau} = 1 - \frac{1}{c} \nabla r \cdot \frac{\partial \mathbf{y}}{\partial \tau} = 1 - \frac{\mathbf{r}}{r} \cdot \frac{\mathbf{v}}{c} = 1 - M_r \quad (31)$$

where  $M_r$  is the Mach number at the point  $\eta$  in the radiation direction at the time  $\tau$ .

The density perturbation can therefore be written as

$$\rho'_S(\mathbf{x}, t) = \int_{S_e} \left[ \frac{Q(\mathbf{y}, \tau)}{4\pi c^2 r |1 - M_r|} \right]_{ret} dS_\eta \equiv \int_{S^*(\tau_e)} \frac{Q^*(\eta, \tau_e)}{4\pi c^2 r_e |1 - M_{r_e}|} dS_\eta \quad (32)$$

Here,  $|1 - M_{r_e}|$  is the *Doppler factor*. The square brackets  $[\ ]_{ret}$  imply that the contents are to be evaluated at the retarded (emission or radiating time)  $\tau_e$  given implicitly by  $g(\tau) = 0$ . The emission position is  $\mathbf{y}_e = \mathbf{y}(\eta, \tau_e)$ , the emission distance  $r_e$  of the source point  $\eta$  to the observer position  $\mathbf{x}$  is  $r_e = |\mathbf{x} - \mathbf{y}(\eta_e, \tau_e)|$ , and  $Q^*(\eta, \tau_e) = Q(\mathbf{y}_e, \tau_e)$ .

### 5.1.1 The thickness noise

The thickness noise is given by the term

$$\tilde{F}_{thickness}(\mathbf{k}, t) = \frac{d}{dt} \int_S \rho_0 u_n e^{-i\mathbf{k} \cdot \mathbf{y}} dS$$

An analysis similar with that of the section 5.1 yields

$$\rho'_{thickness}(\mathbf{x}, t) = \frac{\partial}{\partial t} \int_S \left[ \frac{\rho_0 u_n}{4\pi c^2 r |1 - M_r|} \right]_{ret} dS_\eta \quad (33)$$

### 5.1.2 The loading noise

The last term in relationship (21) describes the loading noise

$$\tilde{F}_{loading}(\mathbf{k}, t) = -ik_j \int_S P'_j e^{-i\mathbf{k} \cdot \mathbf{y}} dS$$

Its contribution to the perturbed density  $\rho'$  is

$$\rho'_{loading}(\mathbf{x}, t) = -\frac{\partial}{\partial x_j} \int_S \left[ \frac{P'_j}{4\pi c^2 r |1 - M_r|} \right]_{ret} dS_\eta \quad (34)$$

We have given here a very short presentation of formulas for thickness noise and loading noise. A more complete presentation about these formulas and their implementation can be found in (Farassat, 2007).

### 5.2 The quadrupole noise term

The last term in formula (21) corresponds to a quadrupole noise source:

$$\begin{aligned} \rho'_q(\mathbf{x}, t) &= \mathcal{F}^{-1} \left\{ ik_j ik_k \tilde{T}_{jk} \right\} \\ &= \int_{-\infty}^t d\tau \int ik_j ik_k \tilde{T}_{jk} \frac{\sin(ck(t-\tau))}{(2\pi)^3 ck} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \\ &= \frac{\partial^2}{\partial x_j \partial x_k} \int_{-\infty}^t d\tau \int_{D^{(e)}_\tau} T_{jk}(\mathbf{y}, \tau) d\mathbf{y} \int \frac{\sin(ck(t-\tau))}{(2\pi)^3 ck} e^{i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})} d\mathbf{k}, \end{aligned} \quad (35)$$

where  $D_\tau^{(e)}$  denotes the 3-D domain occupied by volume sources at the moment  $\tau$ . The last integral in formula (35) was calculated in Appendix B. Introducing its expression given by formula (71) there results

$$\rho'_q(\mathbf{x}, t) = \frac{\partial^2}{\partial x_j \partial x_k} \int_{-\infty}^t d\tau \int_{D_\tau^{(e)}} T_{jk}(\mathbf{y}, \tau) \frac{\delta(\tau - (t - r/c))}{4\pi c^2 r} d\mathbf{y}, \quad (36)$$

the relationship (36) becomes

$$\rho'_q(\mathbf{x}, t) = \frac{\partial^2}{\partial x_j \partial x_k} \int_{D^{(e)}(\tau)} \left[ \frac{T_{jk}(\mathbf{y}, \tau)}{4\pi c^2 r |1 - M_r|} \right]_{ret} d\eta. \quad (37)$$

where the effect of source convection is revealed by the Doppler factor; convection effectively increases the source strength by  $|1 - M_r|^{-1}$ . Further transformations of the formula (37) useful for its numerical implementation were made by Farassat and Brentner (Farassat and Brentner, 1988) and by Brentner in (Brentner, 1997). In the case where the discontinuity surface is permeable (of type  $S_p$ ) this term is missing, the surface being usually chosen outside the space containing the quadruple sources. In the general case the sum of the solutions  $\rho'_q(\mathbf{x}, t)$ ,  $\rho'_{thickness}$  and  $\rho'_{loading}$  completely specify the density field.

## 6. The Kirchhoff method in Aeroacoustics

Besides the Acoustic Analogy approach for the solution of the Aeroacoustic noise, another widely used method is based on Kirchhoffs' solution of the wave equation. We start with the nonhomogeneous wave equation for the pressure perturbation

$$\left[ \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] p'(\mathbf{x}, t) = g(\mathbf{x}, t) \quad (38)$$

where  $g(\mathbf{x}, t)$  represents the density of pressure sources. By applying the Fourier transform with respect to the spatial variables and using formulas (58) and (70) we obtain the operational form of the wave equation (38)

$$\frac{d^2 \tilde{p}'}{dt^2} + c^2 k^2 \tilde{p}' = -c^2 \tilde{g}(\mathbf{k}, t) + \tilde{G}(\mathbf{k}, t) \quad (39)$$

where

$$\begin{aligned} \tilde{G}(\mathbf{k}, t) = & - \int_{S_i} \left( c^2 \left[ \frac{\partial p'}{\partial n} \right] + v_n \left[ \frac{\partial p'}{\partial t} \right] \right) e^{-i\mathbf{k} \cdot \mathbf{y}} dS \\ & - \int_{S_i} c^2 i \mathbf{k} \cdot \mathbf{n} [p'] e^{-i\mathbf{k} \cdot \mathbf{y}} dS - \frac{d}{dt} \int_{S_i} v_n [p'] e^{-i\mathbf{k} \cdot \mathbf{y}} dS \end{aligned} \quad (40)$$

Equation (39) is similar to equation (20). Consequently, its solution can be written as

$$\tilde{p}'(\mathbf{k}, t) = \int_{-\infty}^t \left( -c^2 \tilde{g}(\mathbf{k}, \tau) + \tilde{G}(\mathbf{k}, \tau) \right) \frac{\sin(c k (t - \tau))}{c k} d\tau \quad (41)$$

Hence the contribution of the nonhomogeneous term in equation (38) can be written as

$$\begin{aligned}
 p'_g(\mathbf{x}, t) &= -c^2 \int_{-\infty}^t d\tau \int \tilde{g}(\mathbf{k}, \tau) \frac{\sin(ck(t-\tau))}{(2\pi)^3 ck} e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \\
 &= - \int_{-\infty}^t d\tau \int g(\mathbf{y}, \tau) \frac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c)}{4\pi|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\
 &= - \int g(\mathbf{y}, t-|\mathbf{x}-\mathbf{y}|/c) \frac{d\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}
 \end{aligned} \tag{42}$$

Finally, the contribution of the terms corresponding to the boundary conditions on the mobile surface  $S$  can be written as

$$\begin{aligned}
 p'_G(\mathbf{x}, t) &= - \frac{\partial}{\partial x_i} \int_{S_t} \left[ \frac{p' n_i}{4\pi r |1-M_r|} \right]_{ret} dS \\
 &\quad - \frac{\partial}{\partial t} \int_{S_t} \left[ \frac{p' M_n}{4\pi r |1-M_r|} \right]_{ret} dS \\
 &\quad - \int_{S_t} \left[ \left( \frac{\partial p'}{\partial n} + M_n \frac{\partial p'}{\partial \tau} \right) \frac{1}{4\pi r |1-M_r|} \right]_{ret} dS
 \end{aligned} \tag{43}$$

which coincides with the relationship (5.3) given in (Ffowcs Williams and Hawkins, 1969).

## 7. Concluding remarks

Acoustic Analogy is one of the greatest contributions to the field of acoustics of the previous century. It is a major extension of acoustics made by Sir M. J. Lighthill (and other contributors) who formulated for the first time the science of how sound is created by fluid motion. This theory completes the previous work by famous researchers in the field of acoustics who had discovered how sound propagates through various media and across surrounding surfaces. In this chapter we have attempted to simplify the application of the Acoustic Analogy by showing how to apply it using only classical mathematical analysis tools.

## 8. Acknowledgment

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## 9. Appendix A: Fourier Transform of piecewise differentiable functions

Let  $D^{(i)}(t)$  be a bounded mobile domain in  $\mathbb{R}^3$  having a smooth boundary surface  $S_t$ . Denote by  $D^{(e)}(t)$  the domain external to the surface  $S_t$  (See Fig.2):  $D^{(i)} \cup S \cup D^{(e)} = \mathbb{R}^3$ ,  $D^{(i)} \cap D^{(e)} = \emptyset$ . Let also  $\varphi^{(i)}(\mathbf{x}, t)$  be a continuous differentiable function defined in the closed domain  $\overline{D^{(i)}} \times \mathbb{R}$  and  $\varphi^{(e)}(\mathbf{x}, t)$  a continuous differentiable function defined in  $\overline{D^{(e)}} \times \mathbb{R}$ . We define also

$$\varphi(\mathbf{x}, t) = \begin{cases} \varphi^{(i)}(\mathbf{x}, t), & \text{in } D^{(i)} \\ \varphi^{(e)}(\mathbf{x}, t), & \text{in } D^{(e)} \end{cases} \tag{44}$$

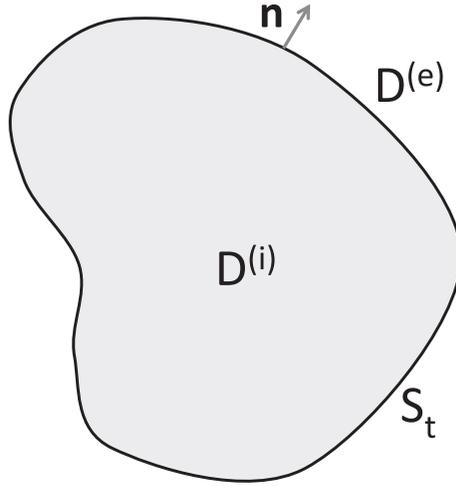


Fig. 2. Boundary Surface

Generally, the function  $\varphi(\mathbf{x}, t)$  is discontinuous across the surface  $S$ . We call it a piecewise differentiable function (pdf). Assuming that the function  $\varphi^{(e)}(\mathbf{x}, t)$  is decreasing sufficiently rapidly at infinity (for more precise conditions about the function  $\varphi^{(e)}(\mathbf{x}, t)$  see (Homencovschi and Singler, 1999)) we can take the Fourier Transform of the function  $\varphi(\mathbf{x}, t)$  with respect to space variables

$$\tilde{\varphi}(\mathbf{k}, t) \equiv \mathcal{F}\{\varphi(\mathbf{x}, t)\} = \int \varphi(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad (45)$$

Here  $\mathbf{x} = (x_1, x_2, x_3)$ ,  $\mathbf{k} = (k_1, k_2, k_3)$ ,  $\mathbf{k} \cdot \mathbf{x} = k_1 x_1 + k_2 x_2 + k_3 x_3$  is the inner product of the two vectors,  $d\mathbf{x} = dx_1 dx_2 dx_3$  and the integral is extended over the whole  $\mathbb{R}^3$  space. The inversion formula can be written as

$$\varphi(\mathbf{x}, t) \equiv \mathcal{F}^{-1}\{\tilde{\varphi}(\mathbf{k}, t)\} = \frac{1}{(2\pi)^3} \int \tilde{\varphi}(\mathbf{k}, t) e^{i\mathbf{x}\cdot\mathbf{k}} d\mathbf{k} \quad (46)$$

where  $d\mathbf{k} = dk_1 dk_2 dk_3$ . Accounting for relationship (44) we can write

$$\mathcal{F}\{\varphi(\mathbf{x}, t)\} = \int_{D(i)} \varphi^{(i)}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \int_{D(e)} \varphi^{(e)}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad (47)$$

### 9.1 Fourier transform of the derivative with respect to a spatial variable of a piecewise differentiable function

#### 9.1.1 The first basic formula

We write

$$\mathcal{F}\left\{\frac{\partial \varphi}{\partial x_1}\right\} = \int_{D(i)} \frac{\partial \varphi^{(i)}(\mathbf{x}, t)}{\partial x_1} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \int_{D(e)} \frac{\partial \varphi^{(e)}(\mathbf{x}, t)}{\partial x_1} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad (48)$$

But

$$\int_{D(i)} \frac{\partial \varphi^{(i)}}{\partial x_1} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = \int_{D(i)} \frac{\partial}{\partial x_1} (\varphi^{(i)} e^{-i\mathbf{k}\cdot\mathbf{x}}) d\mathbf{x} + ik_1 \int_{D(i)} \varphi^{(i)}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad (49)$$

The first integral in the right-hand side of relationship (49) can be replaced, by using the divergence theorem by an integral over the boundary surface  $S_t$ .

$$\int_{D^{(i)}} \frac{\partial}{\partial x_1} \left( \varphi^{(i)} e^{-i\mathbf{k}\cdot\mathbf{x}} \right) d\mathbf{x} = \int_{S_t} n_1 \varphi^{(i)}(\mathbf{y}, t) e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (50)$$

where  $\mathbf{n} = (n_1, n_2, n_3)$  is the external unit normal to  $S_t$ . Therefore,

$$\int_{D^{(i)}} \frac{\partial \varphi^{(i)}}{\partial x_1} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = ik_1 \int_{D^{(i)}} \varphi^{(i)}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \int_{S_t} n_1 \varphi^{(i)}(\mathbf{y}, t) e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (51)$$

Similarly,

$$\int_{D^{(e)}} \frac{\partial \varphi^{(e)}}{\partial x_1} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = ik_1 \int_{D^{(e)}} \varphi^{(e)}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} - \int_{S_t} n_1 \varphi^{(e)}(\mathbf{y}, t) e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (52)$$

Finally, the relationships (49), (51), and (52) give *the first basic formula*:

$$\mathcal{F} \left\{ \frac{\partial \varphi}{\partial x_1} \right\} = ik_1 \tilde{\varphi}(\mathbf{k}, t) - \int_S n_1 [\varphi(\mathbf{y}, t)] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (53)$$

Here we denoted by square bracket the jump of the function  $\varphi(\mathbf{x}, t)$  across the surface  $S_t$

$$[\varphi(\mathbf{y}, t)] = \lim_{\mathbf{x}^{(e)} \rightarrow \mathbf{y}} \varphi^{(e)}(\mathbf{x}^{(e)}, t) - \lim_{\mathbf{x}^{(i)} \rightarrow \mathbf{y}} \varphi^{(i)}(\mathbf{x}^{(i)}, t) \quad (54)$$

for  $\mathbf{y} \in S_t$ ,  $\mathbf{x}^{(i)} \in D^{(i)}$  and  $\mathbf{x}^{(e)} \in D^{(e)}$ . Similar relationships to (53) can be proved for the derivatives in the directions  $x_2$  and  $x_3$ .

**Remark 3.** *It is clear that the obtained relationships can be extended immediately to the case where there are more discontinuity surfaces of the given function. The resulting formulas will contain sums of integrals corresponding to each discontinuity surface.*

### 9.1.2 Other formulas

The relationship (53) gives also

$$\mathcal{F} \{ \nabla \varphi \} \equiv \widetilde{\nabla} \varphi = i\mathbf{k} \tilde{\varphi}(\mathbf{k}, t) - \int_{S_t} \mathbf{n} [\varphi(\mathbf{y}, t)] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (55)$$

In the case where we write  $\mathbf{V}(\mathbf{x}, t) = (V_1(\mathbf{x}, t), V_2(\mathbf{x}, t), V_3(\mathbf{x}, t))$  where  $V_j(\mathbf{x}, t)$  is a piecewise differentiable function defined by a relationship similar to (44) we can write also the formulas

$$\mathcal{F} \{ \nabla \cdot \mathbf{V} \} \equiv \widetilde{\nabla} \cdot \mathbf{V} = i\mathbf{k} \cdot \tilde{\mathbf{V}}(\mathbf{k}, t) - \int_{S_t} \mathbf{n} \cdot [\mathbf{V}(\mathbf{y}, t)] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (56)$$

$$\mathcal{F} \{ \nabla \times \mathbf{V} \} \equiv \widetilde{\nabla} \times \mathbf{V} = i\mathbf{k} \times \tilde{\mathbf{V}}(\mathbf{k}, t) - \int_{S_t} \mathbf{n} \times [\mathbf{V}(\mathbf{y}, t)] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (57)$$

The formulas (55), (56) and (57) permit the calculation of the Fourier Transforms of a gradient of a scalar field of a divergence and a curl of a vector field in the case of piecewise differentiable scalar and vector fields.

Moreover, we can write

$$\begin{aligned}\mathcal{F}\{\Delta\varphi\} &\equiv \mathcal{F}\{\nabla\cdot\nabla\varphi\} = i\mathbf{k}\cdot\widetilde{\nabla}\varphi(\mathbf{k},t) - \int_{S_t} \mathbf{n}\cdot[\nabla(\mathbf{y},t)\varphi]e^{-i\mathbf{k}\cdot\mathbf{y}}dS \\ &= i\mathbf{k}\cdot\left(i\mathbf{k}\widetilde{\varphi}(\mathbf{k},t) - \int_{S_t} \mathbf{n}[\varphi(\mathbf{y},t)]e^{-i\mathbf{k}\cdot\mathbf{y}}dS\right) - \int_{S_t} \left[\frac{\partial\varphi}{\partial n}(\mathbf{y},t)\right]e^{-i\mathbf{k}\cdot\mathbf{y}}dS\end{aligned}\quad (58)$$

Finally,

$$\widetilde{\Delta}\varphi = -k^2\widetilde{\varphi}(\mathbf{k},t) - \int_{S_t} i\mathbf{k}\cdot\mathbf{n}[\varphi(\mathbf{y},t)]e^{-i\mathbf{k}\cdot\mathbf{y}}dS - \int_{S_t} \left[\frac{\partial\varphi}{\partial n}(\mathbf{y},t)\right]e^{-i\mathbf{k}\cdot\mathbf{y}}dS\quad (59)$$

where  $k^2 = k_1^2 + k_2^2 + k_3^2 = |\mathbf{k}|^2$ .

## 9.2 Fourier transform of the time derivative of a piecewise differentiable function

### 9.2.1 The displacement velocity of a mobile surface

Let  $S(y_1, y_2, y_3, t) = 0$  be the equation of the mobile surface  $S_t$ . Then, for  $S(y_{01}, y_{02}, y_{03}, t_0)$  on  $S_{t_0}$  we can write

$$\begin{aligned}0 &= S(y_1, y_2, y_3, t) - S(y_{01}, y_{02}, y_{03}, t_0) = \\ &\left(\frac{\partial S}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial S}{\partial y_2}\frac{dy_2}{dt} + \frac{\partial S}{\partial y_3}\frac{dy_3}{dt} + \frac{\partial S}{\partial t}\right)(t - t_0),\end{aligned}\quad (60)$$

the partial derivatives of the function  $S$  being calculated at a certain point  $\mathbf{x}'$  lying between the points  $\mathbf{x}(t_0)$  and  $\mathbf{x}(t)$ . But,

$$n_1 = \frac{\partial S/\partial y_1}{|\text{grad } S|}, n_2 = \frac{\partial S/\partial y_2}{|\text{grad } S|}, n_3 = \frac{\partial S/\partial y_3}{|\text{grad } S|}\quad (61)$$

are the components of the external normal unit vector  $\mathbf{n}$  and

$$\frac{dy_1}{dt} = v_1, \frac{dy_2}{dt} = v_2, \frac{dy_3}{dt} = v_3,\quad (62)$$

are the projections on the velocity vector of a point on the surface  $S_t$  on the three axes. The relation (60) yields

$$v_n = -\frac{\partial S/\partial t}{|\text{grad } S|}\quad (63)$$

which is the displacement velocity of the geometric surface  $S_t$ . We mention that the displacement velocity of a surface has the direction of the normal vector to this surface.

### 9.2.2 Reynolds' transport theorem

For calculating the Fourier Transform of a time derivative of a certain function we write

$$\begin{aligned}&\int_{D^{(i)}(t)} \varphi^{(i)}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} - \int_{D^{(i)}(t_0)} \varphi^{(i)}(\mathbf{x},t_0)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} = \\ &\int_{D^{(i)}(t)} \varphi^{(i)}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} - \int_{D^{(i)}(t)} \varphi^{(i)}(\mathbf{x},t_0)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} + \\ &\int_{D^{(i)}(t)} \varphi^{(i)}(\mathbf{x},t_0)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} - \int_{D^{(i)}(t_0)} \varphi^{(i)}(\mathbf{x},t_0)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} = \\ &\int_{D^{(i)}(t)} \left\{\varphi^{(i)}(\mathbf{x},t) - \varphi^{(i)}(\mathbf{x},t_0)\right\}e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} + \int_{D^{(i)}(t)-D^{(i)}(t_0)} \varphi^{(i)}(\mathbf{x},t_0)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\end{aligned}\quad (64)$$

Now, dividing by  $(t - t_0)$  and taking the limit for  $t \rightarrow t_0$  the first term gives the Fourier transform of the time derivative of the function  $\varphi^{(i)}(\mathbf{x}, t)$  while in the second integral we can write  $d\mathbf{x} = v_n(t - t_0) dS$  (see (Jacob, 1959), (Currie, 2003)). Finally, we obtain the following form of the Reynolds' transport theorem

$$\begin{aligned} \frac{d}{dt} \int_{D^{(i)}(t)} \varphi^{(i)}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} &= \int_{D^{(i)}(t)} \frac{\partial \varphi^{(i)}(\mathbf{x}, t)}{\partial t} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \\ &+ \int_{S_t} v_n(\mathbf{y}, t) \varphi^{(i)}(\mathbf{y}, t) e^{-i\mathbf{k}\cdot\mathbf{y}} dS \end{aligned} \quad (65)$$

$v_n$  being the displacement velocity of the surface  $S(t)$ .

### 9.2.3 The second basic formula

We calculate now

$$\mathcal{F} \left\{ \frac{\partial \varphi(\mathbf{x}, t)}{\partial t} \right\} = \int_{D^{(i)}(t)} \frac{\partial \varphi^{(i)}(\mathbf{x}, t)}{\partial t} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \int_{D^{(e)}(t)} \frac{\partial \varphi^{(e)}(\mathbf{x}, t)}{\partial t} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$$

By using formula (65) we get

$$\int_{D^{(i)}(t)} \frac{\partial \varphi^{(i)}(\mathbf{x}, t)}{\partial t} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = \frac{d}{dt} \int_{D^{(i)}(t)} \varphi^{(i)} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} - \int_{S_t} v_n \varphi^{(i)}(\mathbf{y}, t) e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (66)$$

$$\int_{D^{(e)}(t)} \frac{\partial \varphi^{(e)}(\mathbf{x}, t)}{\partial t} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = \frac{d}{dt} \int_{D^{(e)}(t)} \varphi^{(e)} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \int_{S_t} v_n \varphi^{(e)}(\mathbf{y}, t) e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (67)$$

The sum of formulas (66) and (67) gives the *second basic formula*

$$\mathcal{F} \left\{ \frac{\partial \varphi(\mathbf{x}, t)}{\partial t} \right\} = \frac{d \widetilde{\varphi}(\mathbf{k}, t)}{dt} + \int_{S_t} v_n(\mathbf{y}, t) [\varphi(\mathbf{y}, t)] e^{-i\mathbf{k}\cdot\mathbf{y}} dS. \quad (68)$$

Formula (68) permits us to calculate the Fourier transform of the time derivative of a piecewise differentiable function. For the second time derivative we can write

$$\mathcal{F} \left\{ \frac{\partial^2 \varphi(\mathbf{x}, t)}{\partial t^2} \right\} = \frac{d}{dt} \frac{d \widetilde{\varphi}}{dt} + \int_{S_t} v_n(\mathbf{y}, t) \left[ \frac{\partial \varphi(\mathbf{y}, t)}{\partial t} \right] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (69)$$

By using again formula (68) we obtain finally,

$$\frac{\widetilde{\partial^2 \varphi}}{\partial t^2} = \frac{d^2 \widetilde{\varphi}}{dt^2} + \frac{d}{dt} \int_{S_t} v_n [\varphi] e^{-i\mathbf{k}\cdot\mathbf{y}} dS + \int_{S_t} v_n \left[ \frac{\partial \varphi}{\partial t} \right] e^{-i\mathbf{k}\cdot\mathbf{y}} dS \quad (70)$$

such that in the Fourier transform of second time derivative of the piecewise differentiable function  $\varphi$  enters the jump of the function  $\varphi$  across the discontinuity and the jump of the first time derivative of  $\varphi$  as well.

## 10. Appendix B: Determination of the Greens' function for the wave equation

By using formula 11, given at page 364 in (Gelfand and Shilov, 1964) we can write

$$\mathcal{F}^{-1} \left\{ \frac{\sin(ckt)}{k} \right\} = \frac{\delta(t - r/c)}{4\pi c r} \quad (71)$$

where  $r = |\mathbf{x}|$ .

By taking the derivative with respect to parameter  $t$  in the both sides of relationship (71) there results

$$\mathcal{F}^{-1} \{ \cos(ckt) \} = \frac{\delta'(t - r/c)}{4\pi c^2 r} \quad (72)$$

Similarly, integrating the formula (71) over the interval  $(0, t)$  we obtain

$$\mathcal{F}^{-1} \left\{ \frac{1 - \cos(ckt)}{k^2} \right\} = \frac{H(t - r/c)}{4\pi r} \quad (73)$$

$H(x)$  being the Heaviside's function.

## 11. References

- Brentner, K. S., An efficient and robust method for predicting helicopter high-speed impulsive noise, *Journal of Sound and Vibration*, 1997, 203(1), 87-100.
- Brentner, K. S. & Farassat, F. 1998 Analytical comparison of the acoustic analogy and Kirchhoff formulation for moving surfaces. *AIAA J.* 36(8), 1998, 1379-1386.
- Crighton, D. G., Dowling, A. P., Ffowcs Williams, J. E., Heckl, M., and Leppington, F. G., *Modern Methods in Analytical Acoustics: Lecture Notes*, Springer-Verlag, London, 1992. Chap.11, Sec. 10.
- Curle, N., The influence of solid boundaries upon aerodynamic sound, *Proceedings of the Royal Society A* 231 (1955) 505-514.
- Currie, I. G., *Fundamental Mechanics of Fluids*, 3rd edition, Marcel Dekker, 2003, pg.12.
- Dowling, A. P., and Ffowcs Williams, J.E., *Sound and Sources of Sound*, Wiley & Sons, New York, 1982. Chap. 9, Sec. 2.
- Farassat, F., Introduction to generalized functions with applications in aerodynamics and aeroacoustics, Corrected Copy (April 1996), NASA Technical Paper 3428, 1994,
- Farassat, F., The Kirchhoff formulas for moving surfaces in aeroacoustics—the subsonic and supersonic cases, NASA Technical Memorandum 110285, 1996.
- Farassat, F., Comments on the paper by Zinoviev and Bies "On acoustic radiation by a rigid object in a fluid flow", *Journal of Sound and Vibration* 281 (2005) 1217-1223.
- Farassat, F., Derivation of Formulations 1 and 1A of Farassat, NASA Technical Memorandum 214853, 2007.
- Farassat, F., and Brentner, K. S., The uses and abuses of the acoustic analogy in helicopter rotor noise prediction, *Journal of the American Helicopter Society*, 1988, 33, 29-36
- Farassat, F., Myers, M.K., Further comments on the paper by Zinoviev and Bies, "On acoustic radiation by a rigid object in a fluid flow", *Journal of Sound and Vibration* 290 (2006), 538-547.
- Ffowcs Williams, J.E., and Hawkings, D.L., Sound generation by turbulence and surfaces in arbitrary motion, *Philosophical Transactions of the Royal Society A* 264 (1969) 321-342.

- Francescantonio, P. Di, A new boundary integral formulation for the prediction of sound radiation, *J. Sound Vibr.* 202 (1997) (4), pp. 491–509.
- Gelfand, I. M., and Shilov, G. E., *Generalized Functions*, Volume 1, Academic Press, New York and London, 1964.
- Goldstein, M. E., *Aeroacoustics*, McGraw-Hill Book Co., 1976.
- Homencovschi, D., and Singler, T., An introduction to BEM by integral transforms, *Engineering Analysis with Boundary Elements*, 23 (1999) 603–609.
- Jacob, C., *Introduction Mathematique a la Mecanique des Fluides*, Gauthier-Villars, Paris, 1959.
- Lighthill, M.J., On sound generated aerodynamically, *Proceedings of the Royal Society of London*, A 211 (1952) 564–586.
- Lighthill, M.J., On sound generated aerodynamically: Turbulence as a source of sound, *Proceedings of the Royal Society of London*, A 222 (1954), 1–32.
- Lyrintzis, A. (1994). Review: The use of Kirchhoff's method in computational aerodynamics, *Transactions of the ASME, Journal of Fluid Engineering* 116(4): 665–676.
- Lyrintzis, A.S., Surface integral methods in computational aeroacoustics.-From the (CFD) near field to the (Acoustic) far-field, *Aeroacoustics*, 2 (2003) pp. 95–128.
- Morino, L. (2003). Is there a difference between aeroacoustics and aerodynamics? An aeroelastician's viewpoint, *AIAA JOURNAL* 41(7): 1209–1223.
- Morris, P.J. and Farassat, F., Acoustic analogy and alternative theories for jet noise prediction, *AIAA Journal*, 40 (2002), pp. 671–680.
- Pilon, A. R., and Lyrintzis, A. S., "Integral Methods for Computational Aeroacoustics," AIAA paper No. 97-0020, presented at the 35th Aerospace Science Meeting, Reno, NV, Jan. 1997.
- Raman, G. (editor), *Computational Aeroacoustics*, Multiscience Publishing Co. Ltd., 2009, 507pp.
- Wu, X-F, and Akay A., Sound radiation from vibrating bodies in motion, *J. Acoust. Soc. Am.*, 91 (1992) pp.2544–2555.
- Zinoviev, A., Bies, D.A., On acoustic radiation by a rigid object in a fluid flow, *Journal of Sound and Vibration* 269 (2004) 535–548.
- Zinoviev, A., Bies, D.A., Author's Reply to: F. Farassat, Comments on the paper by Zinoviev and Bies "On acoustic radiation by arigid object in a fluid flow", *Journal of Sound and Vibration* 281 (2005) 1224–1237.
- Zinoviev, A., Bies, D.A., Author's Reply, *Journal of Sound and Vibration* 290 (2006) 548–554.